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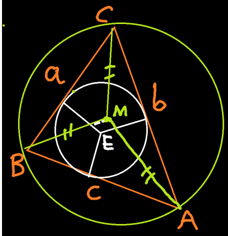
Some basic tools in Geometry

Geometry



- Sine Law : $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$
- cosine Law: $\rightarrow a^2 = b^2 + c^2 - 2bc\cdot\cos A$

$\rightarrow$ if $\hat{A} = 90^\circ \rightsquigarrow\ \Longleftrightarrow\ a^2 = b^2 + c^2$ (Pyth. Thm)

$\rightarrow$ if $\hat{A} < 90^\circ \rightsquigarrow\ \Longleftrightarrow\ a^2 < b^2 + c^2$

$\rightarrow$ if $\hat{A} > 90^\circ \rightsquigarrow\ \Longleftrightarrow\ a^2 > b^2 + c^2$

1) angular bisectors $\rightsquigarrow$ incentre E.
2) perp. bisectors $\rightsquigarrow$ circumcentre M.
3) medians $\rightsquigarrow$ barycentre (centre of gravity)
4) altitudes $\rightsquigarrow$ (orthocentre)

♣ **Today:** lengths, angles, perimeters, areas

If we know the lengths of the 3 sides of a triangle and have no other information what conclusions can we make?

1) We can check if indeed these lengths correspond to the 3 sides of a triangle (which is not always the case)!

The lengths a, b, c correspond to the 3 sides of a triangle if and only if they satisfy the following inequalities: $a < b + c$, $b < a + c$, $c < a + b$.

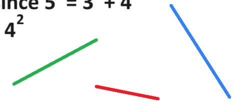
For example, we cannot form a triangle of sides with lengths 3, 2, 1 because we must have $3<2+1$ (which is not true), but we can form a triangle of sides with lengths 3, 4, 6 since $3<4+6$, $4<3+6$, $6<3+4$.

2) We can detect which type of angles we have (and in fact compute them)!

For example, if the lengths are 3, 4, 5 we have a right triangle since $5^2 = 3^2 + 4^2$ and if they are 3, 4, 6 we have an obtuse triangle since $6^2 > 3^2 + 4^2$

3) We can compute the area of the triangle!

To do this we will prove Heron's formula.



Problems

Problem 1.

Consider a triangle whose sides have lengths a, b, and c and let s be its semiperimeter; that is,

$$s = \frac{a+b+c}{2}.$$

Show that its area is given by: $A = \sqrt{s(s-a)(s-b)(s-c)}$.

The above formula is known as Heron's formula.

♣ Question: If two triangles have the same perimeter, do they have the same area?

♦ Answer: No! Consider a right angle triangle of sides 3, 4 and 5 and an equilateral triangle of side 4.

Problem 1: Solution

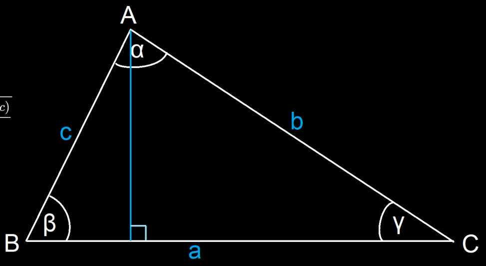
The altitude of the triangle on base a has length $b \sin\gamma$, and it follows

$$A = \frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2}ab\sin\gamma.$$

Applying the law of cosines we get $\cos\gamma = \dfrac{a^2+b^2-c^2}{2ab}$ and so $\sin\gamma = \sqrt{1-\cos^2\gamma} = \dfrac{\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}}{2ab}$.

It follows

$$\begin{aligned}
A &= \frac{1}{4}\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}\\
&= \frac{1}{4}\sqrt{(2ab-(a^2+b^2-c^2))(2ab+(a^2+b^2-c^2))}\\
&= \frac{1}{4}\sqrt{(c^2-(a-b)^2)((a+b)^2-c^2)}\\
&= \sqrt{\frac{(c-(a-b))(c+(a-b))((a+b)-c)((a+b)+c)}{16}}\\
&= \sqrt{\frac{(b+c-a)}{2}\frac{(a+c-b)}{2}\frac{(a+b-c)}{2}\frac{(a+b+c)}{2}}\\
&= \sqrt{\frac{(a+b+c)}{2}\frac{(b+c-a)}{2}\frac{(a+c-b)}{2}\frac{(a+b-c)}{2}}\\
&= \sqrt{s(s-a)(s-b)(s-c)}.
\end{aligned}$$



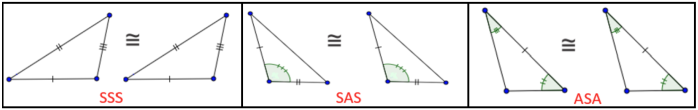
How to compare 2 triangles? (Congruency)

Two triangles are **congruent** if their corresponding sides are equal in length, and their corresponding angles are equal in measure. To check that 2 triangles are congruent it suffices to check that one of the 3 following criteria holds:

(SSS) The three sides are equal.

(SAS) Two angles are the same and the corresponding sides between these angles are the same.

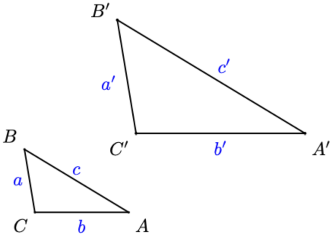
(ASA) Two sides are equal and the corresponding angles between the two sides are equal.



How to compare 2 triangles? (Similarity)

Two triangles are **similar** if they have the same shape but not necessarily the same size.
To check that 2 triangles are similar it suffices to check that one of the 2 following criteria holds:

- The corresponding angles are equal.
- The corresponding sides are proportional.



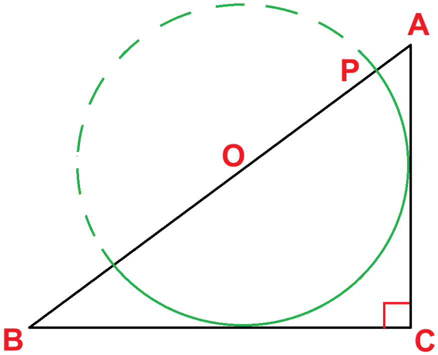
$$\angle A = \angle A'$$
$$\angle B = \angle B'$$
$$\angle C = \angle C'$$

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

♣ Using similar triangles, Thales computed the height of a pyramid!

Problems

Problem 2 (Canada Maths Olympiad).

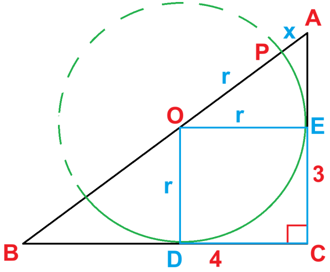


AC=3
BC=4
Compute AP

♦ Problem 2: Solution

from Pythagoras Thm:

$AB = \sqrt{BC^2+AC^2} = \sqrt{16+9} = 5$



AEO similar to ACB

$$\Rightarrow \frac{AO}{OE} = \frac{AB}{BC} \Leftrightarrow \frac{r+x}{r} = \frac{5}{4}$$

$$\Leftrightarrow 4r+4x = 5r \Leftrightarrow \boxed{x = \frac{r}{4}}$$

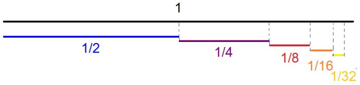
ODB similar to ACB

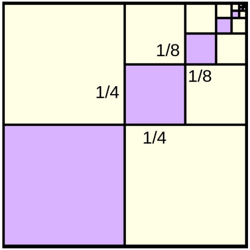
$$\Rightarrow \frac{BD}{DO} = \frac{BC}{CA} \Leftrightarrow \frac{4-r}{r} = \frac{4}{3}$$

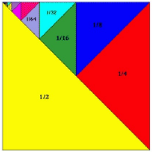
$$\Leftrightarrow 12-3r = 4r \Leftrightarrow \boxed{r = \frac{12}{7}}$$

$$\Rightarrow x = \frac{\frac{12}{7}}{4} \Leftrightarrow \boxed{x = \frac{3}{7}}$$

Euclidean Geometry for infinite summation

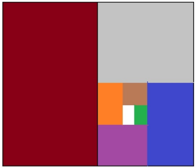




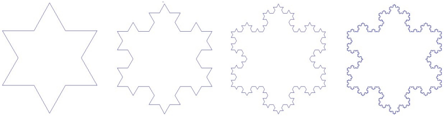


1-1+1-1+1-1+...

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges



$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots =$



♣ Use your geometric intuition to compute the infinite sum $1/2 + 1/4 + 1/8 + 1/16 + \ldots .$ ♦

Basel Problem

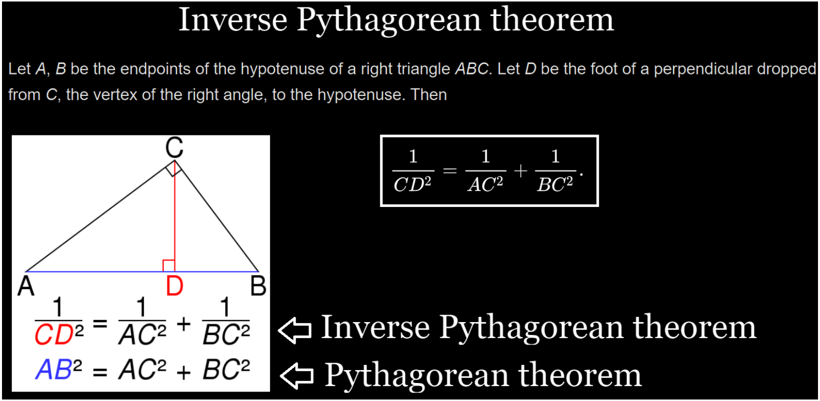
It is hard to compute, or even guess most infinite sums. For example, it turns out that

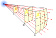
$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

This problem, first posed by Pietro Mengoli in 1650 and solved by Leonhard Euler in 1734(!), is known as the Basel Problem. A (new and much simpler) geometrical solution can be found at:
`https://www.youtube.com/watch?v=d-o3eB9sfls`

The key-ingredient of this new solution is a classical (but less well-known) theorem of Euclidean Geometry, namely the *Inverse Pythagorean theorem*.

Inverse Pythagorean Theorem





This theorem tells us that, if two identical lamps are placed at A and B, then (by the inverse-square law) the amount of light received at C is the same as when a single lamp is placed at D.

♣ Prove the Inverse Pythagorean Theorem.

Inverse Pythagorean Theorem: Proof

Proof: We want to show:

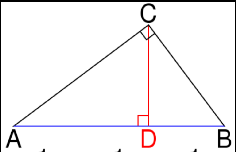
$$\frac{1}{AC^2}+\frac{1}{BC^2}=\frac{1}{CD^2} \iff \frac{BC^2+AC^2}{AC^2\cdot BC^2}=\frac{1}{CD^2}$$

$$\overset{\text{Pyth. Thm}}{\iff} \frac{AB^2}{AC^2\cdot BC^2}=\frac{1}{CD^2} \iff \frac{AB}{AC\cdot BC}=\frac{1}{CD}$$

$$\iff \frac{1}{2}AB\cdot CD=\frac{1}{2}AC\cdot BC,$$

which is true since both sides give the area of $\triangle ABC$.

Thus, $\frac{1}{AC^2}+\frac{1}{BC^2}=\frac{1}{CD^2}$

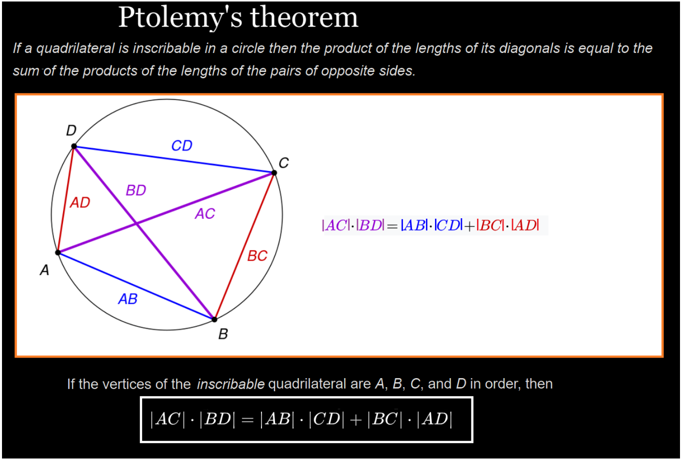


$$\frac{1}{CD^2}=\frac{1}{AC^2}+\frac{1}{BC^2}$$ ← Want to show

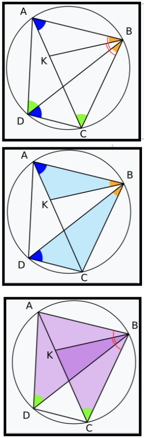
$$AB^2=AC^2+BC^2$$ ← we know (Pythagoras Thm)

Another useful theorem: Ptolemy's Theorem

The following important result (now known as Ptolemy's Theorem) was proved by the Greek mathematician and astronomer Ptolemy for astronomical purposes:



Proof of Ptolemy's Theorem



Goal: $\boxed{AC \cdot BD = AB \cdot CD + BC \cdot DA}$

Let ABCD be a cyclic quadrilateral.

On the chord BC, the inscribed angles $\angle BAC = \angle BDC$, and on AB, $\angle ADB = \angle ACB$.

Construct K on AC such that $\angle ABK = \angle CBD$;

since $\angle ABK + \angle CBK = \angle ABC = \angle CBD + \angle ABD$, $\boxed{\angle CBK = \angle ABD}$

$\Rightarrow$ $\triangle ABK$ is similar to $\triangle DBC$; $\triangle ABD$ is similar to $\triangle KBC$.

Thus $AK/AB = CD/BD$, and $CK/BC = DA/BD$

$\Rightarrow$ $\boxed{AK \cdot BD = AB \cdot CD, \text{ and } CK \cdot BD = BC \cdot DA}$ $\overset{+}{\Rightarrow}$

$AK \cdot BD + CK \cdot BD = AB \cdot CD + BC \cdot DA$ $\Rightarrow$

$$\left\{ \begin{array}{l} (AK+CK) \cdot BD = AB \cdot CD + BC \cdot DA. \\ AK + CK = AC \end{array} \right\} \Rightarrow \boxed{AC \cdot BD = AB \cdot CD + BC \cdot DA}$$

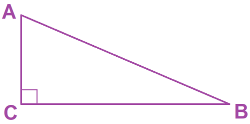
♣ For a visual proof see:
https://en.wikipedia.org/wiki/Ptolemy%27s_theorem

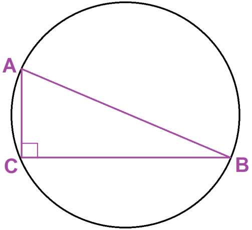
Applications of Ptolemy's Theorem

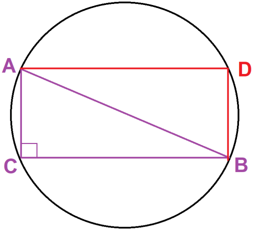
- Prove Pythagoras' Theorem using Ptolemy's Theorem.
- Given an equilateral triangle inscribed on a circle and a point Q on the circle, show that the distance from the point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.
- Given a regular pentagon which has side length a and (common) distance b of any two non-adjacent vertices, show that
  $\frac{b}{a} = \frac{1+\sqrt{5}}{2} =: \phi$ (golden ratio).

♦ Ptolemy's Theorem implies Pythagoras' Theorem

Let ABC be a right triangle. Consider a circle such that the vertices A, B and C lie on that circle.







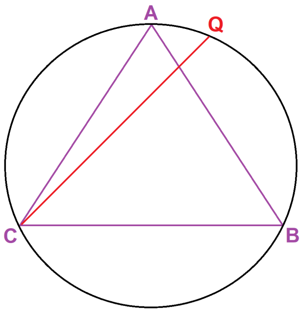
Consider the unique point D on the circle such that AD is parallel to CB and BD is parallel to CA. Thus, ACBD is a rectangle. By applying Ptolemy's Theorem for the cyclic quadrilateral ACBD, we get that

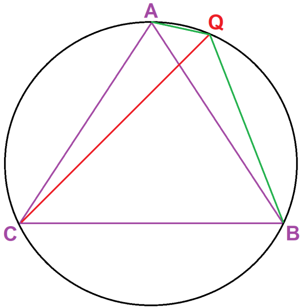
$$AB \cdot CD = AD \cdot CB + CA \cdot BD.$$

Since CD=AB, AD=CB and BD=CA (because ACBD is a rectangle), we conclude that

$$AB^2 = CB^2 + CA^2 \qquad \text{(Pythagoras' Theorem).}$$

♦ Ptolemy's Theorem for equilateral triangles





Let ABC be an equilateral triangle inscribed on a circle and Q a point on this circle. By applying Ptolemy's Theorem to the cyclic quadrilateral AQBC we get that

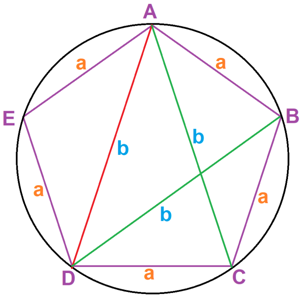
$$AB \cdot QC = AC \cdot QB + CB \cdot AQ\,.$$

Since AB=AC=CB, we conclude that

$$QC = QB + AQ\,,$$

that is, the distance from the point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.

Ptolemy's Theorem for regular pentagons



Let ABCDE be a regular pentagon which has side length a and (common) distance b of any two non-adjacent vertices.
By applying Ptolemy's Theorem to the cyclic quadrilateral ABCD, we get

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

This gives $b \cdot b = a \cdot a + a \cdot b$, or equivalently, $b^2 - a \cdot b - a^2 = 0$.
If we solve for b we get the following two solutions:

$$b = \frac{a \pm \sqrt{5a^2}}{2} = \frac{1 \pm \sqrt{5}}{2} \cdot a.$$

Since b is a length we accept only the positive solution, and so we conlcude that

$$\frac{b}{a} = \frac{1 + \sqrt{5}}{2} \quad \text{(golden ratio).}$$

Problems

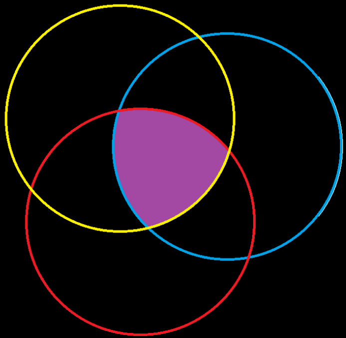
Problem 3.

(i) Find a shape (with no holes) that has constant width, but is **not** a circle.

(ii) For the shape you found in part (i), compute its perimeter as a function of its (constant) width w. What do you observe?

(iii) For the shape you found in part (i), compute its area as a function of its (constant) width w.

♦ Problem 3: Solution

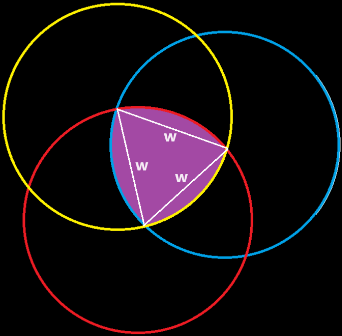


(i) Consider the purple shape, which the intersection of 3 discs of equal radii which have the following property: the centre of each disc coincides with one (of the 2) intersection points of the other 2 discs. It is easy to see that the width of each shape is constant (in fact it is equal to the radius of the discs).

(ii) The 3 vertices of the purple shape (called Reuleaux triangle) form an equilateral triangle. Thus, each side is an arc of 60°, and so its length is 60/360 times the perimeter of each circle which is $2\pi w$. Hence the perimeter of the Reuleaux triangle is πw.

We observe that the Reuleaux triangle has the same perimeter with the circle of the same width! This is not a coincidence: Barbier's Theorem tells us that all shapes of constant width w have perimeter πw!

♦ Problem 3: Solution



(iii) Since the area of each meniscus-shaped portion of the Reuleaux triangle is a circular arc with opening angle 60 , we get that its area (let's call it M) equals the area of the cyclic sector minus the area of the equilateral triangle. Thus, we have

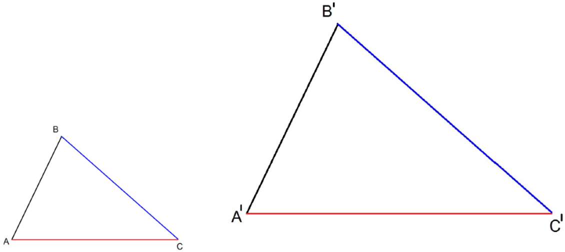
$$M = (60/360)\pi w^2 - (\sqrt{3}/4)w^2$$

Hence the total area A of the Reuleaux triangle is equal to 3M minus the area of the equilateral triangle:

$$A = 3M - (\sqrt{3}/4)w^2 = \frac{(\pi - \sqrt{3})w^2}{2}$$

In fact, the Reuleaux triangle has the smallest area for a given width of any curve of constant width w!

Areas of similar triangles



Let ABC and $A'B'C'$ be two similar triangles, that is,

$$\frac{A'B'}{AB} = \frac{C'A'}{CA} = \frac{B'C'}{BC} = \text{ratio of similarity}$$

Then

$$\frac{[A'B'C']}{[ABC]} = \left(\frac{A'B'}{AB}\right)^2 = \left(\frac{C'A'}{CA}\right)^2 = \left(\frac{B'C'}{BC}\right)^2.$$

Proposition. The ratio of areas of two similar triangles equals the square of ratio of similarity.

Problems

Problem 4. Let Q be a point inside a triangle ABC. Three lines pass through Q and are parallel with the sides of the triangle. These lines divide the initial triangle into six parts, three of which are triangles of areas S_1, S_2 and S_3. Prove that

$$\sqrt{[ABC]} = \sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}.$$

Problem 4: Solution

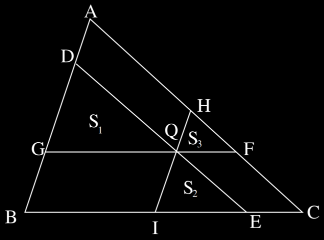
Let D, E, F, G, H, I be the points of intersection between the three lines and the sides of the triangle.

Then triangles DGQ, HQF, QIE and ABC are similar so

$$\frac{S_1}{[ABC]} = \left(\frac{GQ}{BC}\right)^2 = \left(\frac{BI}{BC}\right)^2$$

Similarly

$$\frac{S_2}{[ABC]} = \left(\frac{IE}{BC}\right)^2, \quad \frac{S_3}{[ABC]} = \left(\frac{QF}{BC}\right)^2 = \left(\frac{CE}{BC}\right)^2.$$



Then

$$\sqrt{\frac{S_1}{[ABC]}} + \sqrt{\frac{S_2}{[ABC]}} + \sqrt{\frac{S_3}{[ABC]}} = \frac{BI}{BC} + \frac{IE}{BC} + \frac{EC}{BC} = 1.$$

This yields

$$\sqrt{[ABC]} = \sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}.$$

Some useful/interesting links:

- https://imogeometry.blogspot.com/p/1.html
- http://www.imo-official.org/problems.aspx
- https://www.geogebra.org/t/geometry
- https://thatsmaths.com/tag/geometry/
- https://www.ucd.ie/mathstat/newsandevents/events/mathsenrichment/
- https://artofproblemsolving.com/wiki/index.php/Resources_for_mathematics_competitions
- https://en.wikipedia.org/wiki/Reuleaux_triangle
- https://www.youtube.com/watch?v=cUCSSJwO3GU&t=287s&ab_channel=Numberphile